Written Homework #5

① $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = I_2 = \begin{bmatrix} a+c+e & b+d+f \\ 2a+c+4e & 2b+d+4f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$a+c+e=1$ | $4a+4c+4e=4$ | $a=2$ | $4b+4d+4f=0$ | $d=0$

$b+d+f=0$ | $-\ 2a+c+4e=0$ | $e=-1$ | $-\ 2b+d+4f=1$ | $b=-\frac{1}{2}$

$2a+c+4e=0$ | $2a+3c+0=4$ | $c=0$ | $2b+3d+0=-1$ | $f=\frac{1}{2}$

$2b+d+4f=1$ | $a=\frac{4-3c}{2}$ | | $b=-\frac{1-3d}{2}$

$\boxed{B = \begin{bmatrix} 2 & -1/2 \\ 0 & 1/2 \\ -1 & 1/2 \end{bmatrix}}$

② $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

a) $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$a^2+bc=0$ | $a=1$

$ab+bd=0$ | $b=1$ | $\boxed{A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}}$

$ca+dc=0$ | $c=-1$

$cb+d^2=0$ | $d=-1$

b) $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$a^2+bc=a \Rightarrow (1-c)^2+bc=1-c \Rightarrow c^2+bc=c \Rightarrow b=1-c \Rightarrow b=a$

$ab+bd=b \Rightarrow a=1-d$ | $a=0, b=0$ | $\boxed{A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}$

$ca+dc=c \Rightarrow a=1-d$ | $c=1, d=1$

$cb+d^2=d \Rightarrow$

$c=1$ c) $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$a^2+bc=1$ | $a=-d \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ $\boxed{A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$

$ab+bd=0$

$ca+ac=0$

$cb+a^2=1$

③ (a) $Av=b$, $Aw=b$ $\quad A(cv+(1-c)w) = cAv+(1-c)Aw = cb+(1-c)b=b$

so for any value of c, $cv+(1-c)w$ is a solution for $Ax=b$, which means $Ax=b$ has ∞ solutions.

(b) $Ax=0 \Rightarrow$ since $Av=b$ and $Aw=b$, then $Ax=Av-Aw \Rightarrow A(v-w)$

$v-w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -4 \\ 5 \\ 2 \end{bmatrix}}$

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(4) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & 3 & -2 \\ 1 & -2 & 2 \\ 2 & -6 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

a) The three conditions of a subspace are:

i) $\vec{0}$ is in S

ii) If $\vec{u}$ & $\vec{v}$ are in S, then $\vec{u}+\vec{v}$ is in S

iii) if $\vec{u}$ is in S and c is a scalar, $c\vec{u}$ is in S

Condition i is met because $T(\vec{0}) = \vec{0}$, so $\vec{0} \in V$

Condition ii is met because $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) = \vec{u}+\vec{v}$, so $\vec{u}+\vec{v} \in V$

Condition iii is met because $T(c\vec{v}) = cT(\vec{v}) = c\vec{v}$, so $c\vec{v} \in V$

Therefore, V must be a subspace of $\mathbb{R}^3$

b) $\begin{bmatrix} 0 & 3 & -2 \\ 1 & -2 & 2 \\ 2 & -6 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Longrightarrow \begin{bmatrix} 3y - 2z - x \\ x - 3y + 2z \\ 2x - 6y + 4z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ All 3 equations are equivalent to the first one, so: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V \Leftrightarrow 3y - 2z - x = 0$

c) $3y - 2z - x = 0$ is the equation of a plane that passes through the origin, so V is a plane.

d) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V \Leftrightarrow 3y - 2z - x = 0 \Longrightarrow x = 3y - 2z \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y-2z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

So, the basis for V is: $\left(\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right)$

(5) The vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ can be part of a basis for V because any vectors can be substituted into the equation $\vec{w}+\vec{x}+\vec{y}+\vec{z}=0$ if they are linear combinations of $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. To expand the set of these two vectors to a basis of V, I can add a linearly independent vector, for example $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Therefore the basis for the subspace V is given by:

$\boxed{\left(\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)}$

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(1) Let
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}.$$
Find a 3×2 matrix B with $AB = I_2$. Is there more than one matrix B with this property? Justify your answer.

(2) Find a 2×2 matrix A, which is not the zero or identity matrix, satisfying each of the following equations.
a) $A^2 = 0$
b) $A^2 = A$
c) $A^2 = I_2$

(3) Suppose that A is a matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^2$. Suppose further that $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$ are both solutions to the equation $A\mathbf{x} = \mathbf{b}$.
(a) How many solutions does the equation $A\mathbf{x} = \mathbf{b}$ have? Explain your answer.
(b) Find a nontrivial solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Justify that it is indeed a solution.

(4) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by
$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & 3 & -2 \\ 1 & -2 & 2 \\ 2 & -6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$
Let $\mathcal{V}$ be the set of all vectors that are fixed by T, which means that $\mathcal{V} = \{\mathbf{v} \in \mathbb{R}^3 \ : \ T(\mathbf{v}) = \mathbf{v}\}$.
(a) Show, using the definition of subspace, that $\mathcal{V}$ is a subspace of $\mathbb{R}^3$.
(b) Come up with an equation that also defines $\mathcal{V}$. (In other words, find a linear equation $ax + by + cz = d$ such that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathcal{V} \ \Leftrightarrow \ ax + by + cz = d$.)
(c) Geometrically, what kind of object is $\mathcal{V}$ (point/line/plane etc)?
(d) Find a basis for $\mathcal{V}$.

(5) Let V be the subspace of $\mathbb{R}^4$ defined as
$$V = \{(w, x, y, z) \in \mathbb{R}^4 \ : \ w + x + y + z = 0\}.$$
Check that the vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ can be part of a basis for V. Then expand the set consisting of these two vectors to a basis of V.